Roundoff Error and the Patriot Missile

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The March 13 issue of Science carried an article claiming, on the basis of a report from the General Accounting Office (GAO), that a "minute mathematical error ... allowed an Iraqi Scud missile to slip through Patriot missile defenses a year ago and hit U.S. Army barracks in Dhahran, Saudi Arabia, killing 28 servicemen." The article continues with a readable account of what happened.

The article says that the computer doing the tracking calculations had an internal clock whose values were slightly truncated when converted to floating-point arithmetic. The errors were proportional to the time on the clock: 0.0275 seconds after eight hours and 0.3433 seconds after 100 hours. A calculation shows each of these relative errors to be both very nearly 2^{-20}, which is approximately 0.0001%.

The GAO report contains some additional information. The internal clock kept time as an integer value in units of tenths of a second, and the computer's registers were only 24 bits long. This and the consistency in the time lags suggested that the error was caused by a fixed-point 24-bit representation of 0.1 in base 2. The base 2 representation of 0.1 is nonterminating; for the first 23 binary digits after the binary point, the value is $0.1 \times (1 - 2^{-20})$. The use of $0.1 \times (1 - 2^{-20})$ in obtaining a floating-point value of time in seconds would cause all times to be reduced by 0.0001%.

This does not really explain the tracking errors, however, because the tracking of a missile should depend not on the absolute clock-time but rather on the time that elapsed between two different radar pulses. And because of the consistency of the errors, this time difference should be in error by only 0.0001%, a truly insignificant amount.

Further inquiries cleared up the mystery. It turns out that the hypothesis concerning the truncated binary representation of 0.1 was essentially correct. A 24-bit representation of 0.1 was used to multiply the clock-time, yielding a result in a pair of 24-bit registers. This was transformed into a 48-bit floating-point number. The software used had been written in assembly language 20 years ago. When Patriot systems were brought into the Gulf conflict, the software was modified (several times) to cope with the high speed of ballistic missiles, for which the system was not originally designed.

At least one of these software modifications was the introduction of a subroutine for converting clock-time more accurately into floating-point. This calculation was needed in about half a dozen places in the program, but the call to the subroutine was not inserted at every point where it was needed. Hence, with a less accurate truncated time of one radar pulse being subtracted from a more accurate time of another radar pulse, the error no longer cancelled.

In the case of the Dhahran Scud, the clock had run up a time of 100 hours, so the calculated elapsed time was too long by $2^{-20} \times 100$ hours = 0.3433 seconds, during which time a Scud would be expected to travel more than half a kilometer.

The roundoff error, of course, is not the only problem that has been identified: serious doubts have been expressed about the ability of Patriot missiles to hit Scuds.

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